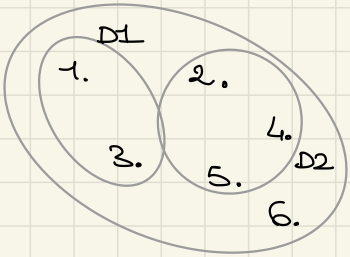
# FONDAMENTI DI SEGNALI E TRASMISSIONI

# PROBABILITÀ

Esperimenti $\longrightarrow$ eventi




- semplici $\{1, 2, 3, 4, 5, 6\}$
- complessi D1 D2

UNIONE o
$+$
$\cup$

INTERSEZIONE e
,
$\cap$

Probabilità :

- Frequentistica

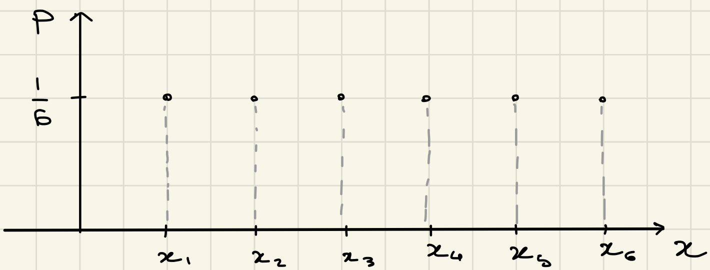
A evento
N esperimenti

$$\boxed{P(A) = \frac{N_A}{N}}$$

$$\boxed{N \to +\infty}$$

in un dado:

$$P(\text{"1"}) = \frac{1}{6}$$



- Assiomatica

\# $P \geq 0$

\# $P \leq 1$

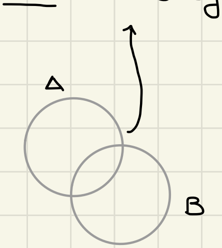
\# $P(x_i + x_j) = P(x_i) + P(x_j)$

solo se $x_i$ e $x_j$ sono disgiunti

cioè $x_i \cap x_j = \emptyset$

$$\boxed{\begin{gathered} P(x_1 + x_2 + \ldots + x_n) = 1 \\ \sum_{i=1}^{n} P(x_i) = 1 \end{gathered}}$$

non sono disg.




$$P(A+B) = P(A) + P(B) + P(A,B)$$

in un dado:

A $\quad x \leq 4$

B $\quad x \geq 3$

| $x_i$ | P |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

$P(A) = \frac{4}{6} = \frac{2}{3}$

$P(B) = \frac{4}{6} = \frac{2}{3}$

$P(A+B) = \frac{4}{6} + \frac{4}{6} - \frac{2}{6} \rightarrow x_i = 3, 4$

$= 1$

(calcolo combinatorio)

faccio tante prove e vedo che succede

# Metodo Montecarlo (Matlab)

Codice:

```
% calcolo prob.
ntry = 100;
x = randi(6, 1, ntry)
```

commento

; → non stampare a schermo il valore

ntry → lunghezza vettore

6 → valore casuale da 1 a 6

1 → dimensione matrice (monodimensionale)

```
p = (x <= 4) || (x >= 3)
Nwin = sum(p)
```

## PROBABILITÀ CONDIZIONATE e DIPENDENZA STATISTICA

Pioggia oggi — eventi (P0, P1, P2) — P0 no pioggia, P2 nubifragio

Cielo ieri — (C0, C1) — C0 sereno, C1 nuvoloso

$$P(P2|_{C1}) \neq P(P2)$$

condizionato

| cielo | C0 | C0 | C0 | C1 | ... | C1 | C1 |
|---|---|---|---|---|---|---|---|
| pioggia | P0 | P0 | P2 | P1 | ... | P2 | P2 |
| | 1 | 2 | 3 | 4 | ... | N-1 | N |

(N grande)

$$P(P2|_{C1}) = \frac{2}{3} \gg P(P2) = \frac{2}{N}$$

le probabilità che piova aumentano se il giorno prima era nuvoloso

$$P(P2|_{C1}) = \frac{N(P2, C1)}{N(C1)} = \frac{N(P2, C1)/N}{N(C1)/N} = \frac{P(P2, C1)}{P(C1)}$$

$$\Longrightarrow \boxed{P(A|_B) = \frac{P(A, B)}{P(B)}}$$

Se A e B sono TOTALMENTE DIPENDENTI

$$\underline{P(A|_B) = 1}$$

Se A e B sono TOTALMENTE INDIPENDENTI

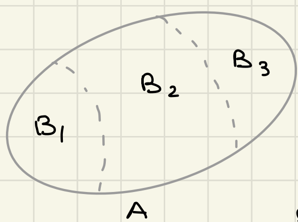
$$\underline{P(A|_B) = P(A)}$$

inoltre $\dfrac{P(A,B)}{P(B)} = P(A) \Rightarrow \underline{P(A,B) = P(A)P(B)}$
$= P(A|_B)P(B)$

$$P(B,A) = P(A,B)$$
$$P(B|_A)P(A) = P(A|_B)P(B) \Rightarrow \boxed{P(B|_A) = P(A|_B)\frac{P(B)}{P(A)}}$$

Regola di Bayes

## Probabilità totale




$P(A) = ?$ Scompongo l'evento A in tanti eventi più semplici $B_1, B_2, B_3$ disgiunti. $A = B_1 \cup B_2 \cup B_3$

$$B_i \cap B_J = \emptyset \quad \forall i \neq J$$

$$\boxed{P(A) = \sum_n P(A, B_n)} = \sum_n P(A|_{B_n})P(B_n)$$

es: il dado ha memoria?
D1 = "6" 1° dado
D2 = "6" 2° dado
Se $P(D1, D2) = P(D1)P(D2)$ i due eventi sono disgiunti $\rightarrow$ il dado non ha memoria.

$$1/36 = 1/6 \cdot 1/6 \quad \checkmark$$

es: gioco delle quattro carte

$\underset{I}{\boxed{G}}$ $\underset{II}{\boxed{G}}$ $\underset{III}{\boxed{G}}$ $\underset{IV}{\boxed{R}}$ $\qquad P(II_R | I_G) = \frac{1}{3} \xrightarrow{\neq} P(II_R) = \frac{1}{4}$

$\Rightarrow$ l' evento dipende dal conoscere il colore della $I^a$ carta

Calcolare la probabilità di $P(I_G, II_R)$:

- calcolo combinatorio

$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ permutazioni possibili

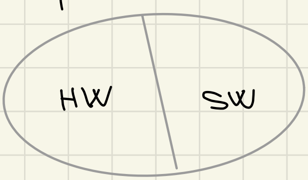
$N(I_G, II_R) = 3 \cdot 1 \cdot 2 = 6$

$\hookrightarrow P(I_G, II_R) = \frac{6}{24} = \frac{1}{4}$

- metodo montecarlo

```
ntry = 10000;
nf = 0;                          nf = P(I_G, II_R)
for    i = 1 : ntry
       t = randperm (4,2);
       if( t(1) == 1   &   t(2) == 2)
              nf = nf + 1
end
```

es: probabilità di guasto




sistema complesso

$p_H$ probab. HardWare guasto

$p_S$ " SoftWare "

Probab. funzionamento: $P((1-P_H);(1-P_S)) \underset{\text{indipend.}}{=} (1-P_H)\cdot(1-P_S)$

Probab. guasto: $1-(1-P_H)(1-P_S) =$
$= P_H + P_S - P_H P_S$ $\simeq \emptyset$ se le probab. sono molto piccole

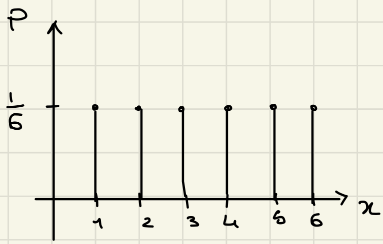
Se ho n componenti: $\boxed{P_f = (1-P_n)^n}$
$P_g = 1 - P_f$

## DENSITÀ e RIPARTIZIONI

esperimento $\rightarrow$ risultati eventi $\rightarrow$ probabilità

In un dado:

DISCRETO



DISTRIBUZIONE UNIFORME

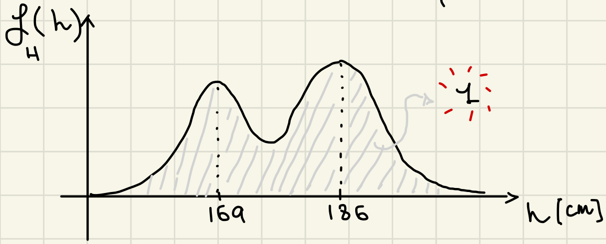
DENSITÀ di PROBABILITÀ
$P(x_i) \geq 0$ $\quad x = \text{randi}(6,1,1)$
$\sum_i P(x_i) = 1$

V.C. variabile casuale

distrib. uniforme

Altezza delle persone:

CONTINUO



$h = 157,31311\ldots$ cm

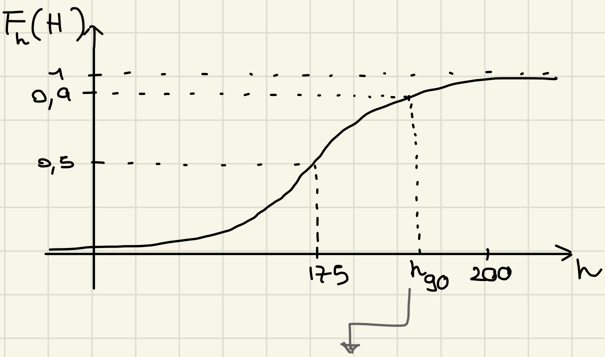
$h_i$ — la prob. di un punto è 0

$$f_h(H) = \frac{P(h_i \leq H < h_i + dh)}{dh} \geq 0$$

$$P(h < 180) = \int_{-\infty}^{180} \underbrace{f_H(h)\, dh}_{P(\sim h)}$$

$$\int_{-\infty}^{+\infty} f_H(h)\, dh = 1$$

## PERCENTILI




90esimo percentile

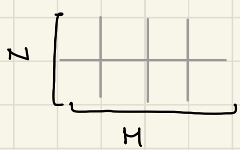
$h_{50}$ = MEDIANA

**PDF** o ddp
Probability Density Function

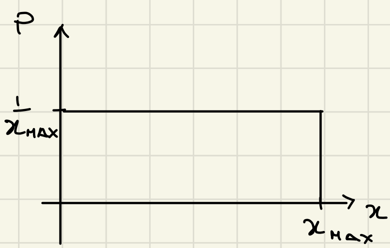
$$f_H(h) \underset{d/dh}{\overset{\int_{-\infty}^{h} dh}{\rightleftarrows}} F_H(h)$$

## DISTRIBUZIONE UNIFORME


$x = \text{randi}(6, N, M) \qquad x \in [1, 6]$




$x = \text{rand}(N, M) \qquad x \in [0, 1]$




argomento ddp

$$\left[ f_X(x) = \frac{1}{6}\delta(x-1) + \frac{1}{6}\delta(x-2) + \dots \right.$$
$$\left. \dots + \frac{1}{6}\delta(x-6) \right]$$

V.C.

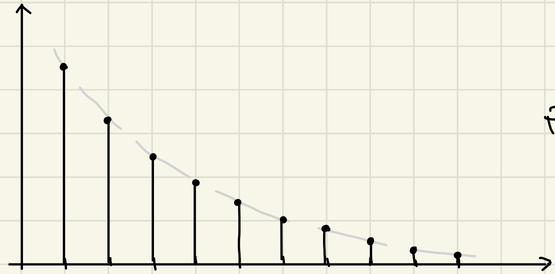
lancio di un dado in forma continua

# DISTRIBUZIONE GEOMETRICA

Prob. 1 successo prova N $\quad N=1 \quad N=2 \quad N=3$

$$p \qquad qp \qquad q^2 p$$

PDF

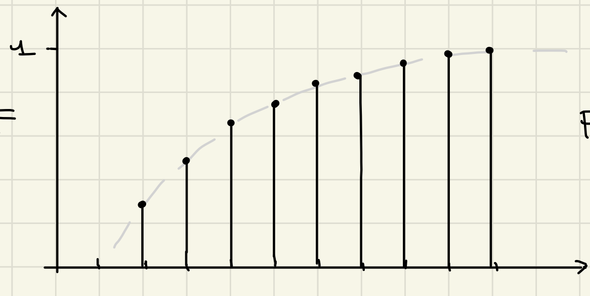



$$P_N(n) = q^{n-1} p$$

probabilità di successo $\boxed{p}$

insuccesso $\boxed{q = 1 - p}$

CDF

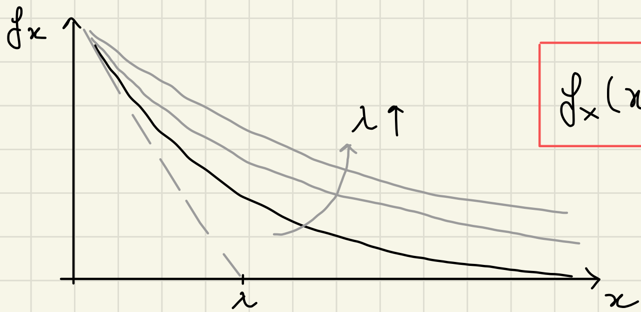



$P_N(N \leq n)$

CDF
Cumulative
Distribution
Function

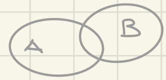
$$\text{CDF}: \quad \sum_{n=1}^{N} p q^{n-1} = p \sum_{n=1}^{N} q^{n-1} = p \, \frac{1-q^N}{1-q} \xrightarrow{N \to +\infty} 1$$

Gli eventi di una distribuzione geometrica sono <u>indipendenti</u> $\longrightarrow$ no memoria




$$\boxed{f_x(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} u(x)}$$

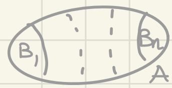
# Ripasso:




- $P(A+B) = P(A) + P(B) - P(A,B)$
- $P(A,B) = P(A|_B)\,P(B) = P(A)\,P(B)$

  ↑ se sono indipendenti

- $P(A) = \sum_{i=1}^{n} P(A, B_i)$




  $= \sum_{i=1}^{n} P(A|_{B_i})\,P(B_i)$ prob. totale

- $P(A|_B)\,P(B) = P(A,B) = P(B,A) = P(B|_A)\,P(A)$

$$P(A|_B) = P(B|_A)\,\frac{P(A)}{P(B)}$$ regola di Bayes

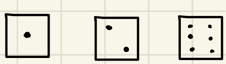
A cosa serve la Probabilità?

* Prendere decisioni

* Fare previsioni

* Valutare rischi e benefici

es: gioco del "6"



3 dadi

"6" ⟶ + 1€

"66" ⟶ + 2€

"666" ⟶ + 3€

altrimenti ⟶ − 1€

$6^3 = 216$ combinazioni

$\quad\;\; \| $

$\quad\;\; N$

## calcolo combinatorio

1 1 1
1 1 2
· · · $-5€$
1 1 6 $\longrightarrow$ $+1€$
1 2 1 5
· · · $-5€$
1 2 6 $\longrightarrow$ $+1€$
· · ·
· · · $-15€$, $+3€$
1 6 1 $\longrightarrow$ $+1€$
1 6 2 $\longrightarrow$ $+1€$ 6
· · ·
1 6 6 $\longrightarrow$ $+2€$
2 1 1
2 1 2
· · ·
5 5 6 $\longrightarrow$ $+1€$ $11 \times 4 = 44$
· · ·
5 6 6 $\longrightarrow$ $+2€$
6 1 1 $\longrightarrow$ $+1€$
· · · $6 \times 6 = 36$
6 6 6 $\longrightarrow$ $+3€$

$$36 + 44 + 11 = \underline{91}$$

Probabilità che esca <u>almeno un "6"</u>:

$$P(\text{"6"}) = \frac{N(\text{"6"})}{N} = \frac{91}{216} \simeq 0{,}4213$$

formule

in un solo dado $P(\text{"}\overline{6}\text{"}) = \frac{5}{6}$ — not "6" = "1", "2", "3", "4", "5"

in 3 dadi $P(\text{"}\overline{6}\text{"}\,\text{"}\overline{6}\text{"}\,\text{"}\overline{6}\text{"}) = \left(\frac{5}{6}\right)^3 = \frac{125}{216} = Q$

probab. che non esca nemmeno un "6"

$\Longrightarrow$ probab. che esca almeno un "6":

$$P = 1 - Q = 1 - \frac{125}{216} = \frac{91}{216}$$

es.

R R B B V V V
I II III IV V VI VII

palline colorate
2B, 2R, 3V

$$P(IV, IIB, IIIB) = ?$$

$$P(A, B) = P(A|_B)\,P(B)$$

$$P(IV, IIB) = \underbrace{P(IV|_{IIB})}_{\frac{3}{6} = \frac{1}{2}}\ \underbrace{P(IIB)}_{\frac{2}{7}} = \frac{1}{7}$$

$$P(IV, IIB, IIIB) = P(IIIB, IV, IIB) =$$
$$= \underbrace{P(IIIB|_{IV,IIB})}_{\frac{1}{5}}\ \underbrace{P(IV, IIB)}_{\frac{1}{7}} = \frac{1}{35}$$

Se gli eventi fossero stati indipendenti:

$$P(IV, IIB, IIIB) = P(IV)\,P(IIB)\,P(IIIB) = \frac{3}{7}\cdot\frac{2}{7}\cdot\frac{2}{7} =$$
$$= \frac{12}{243} \neq \frac{1}{35}$$

es: test clinico

malattia rara $1/100000 = 10^{-5}$

test identifica la malattia $99,9\% = 0,999$

test identifica un sano come malato $0,01\% = 0,0001$

| | $T_s$ | $T_M$ |
|---|---|---|
| $S_s$ | $1-10^{-4}$ | $10^{-4}$ |
| $S_M$ | $10^{-3}$ | $0,999$ |

$T \rightarrow$ Test

$S \rightarrow$ Soggetto

ERRORI: se cerco di diminuire uno cresce l'altro

$\rightarrow$ matrice di confusione

$$P(S_s|T_M) = ?$$
$$= \underbrace{P(T_M|S_s)}_{10^{-4}} \frac{\overbrace{P(S_s)}^{1-10^{-5}}}{P(T_M)} = \frac{10^{-4} \cdot 0,9999}{1,1 \cdot 10^{-4}} \simeq 0,91$$

$$P(T_M) = P(T_M|S_s)P(S_s) + P(T_M|S_M)P(S_M) =$$
$$= 10^{-4}(1-10^{-5}) + 0,999 \cdot 10^{-5} \simeq 1,1 \cdot 10^{-4}$$

## DENSITÀ e RIPARTIZIONI in casi PLURIDIMENSIONALI

2 dadi a 3 facce:

$$P(\underbrace{D1 = 1}_{x}, \underbrace{D1 + D2 = 2}_{z})$$

| $z$ \ $x$ | 1 | 2 | 3 |
|---|---|---|---|
| 6 | 0 | 0 | 1/3 |
| 5 | 0 | 1/3 | 1/3 |
| 4 | 1/3 | 1/3 | 1/3 |
| 3 | 1/3 | 1/3 | 0 |
| 2 | 1/3 | 0 | 0 |

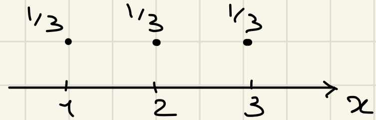
$$P(x_i, z_j) = P(X = x_i, Z = z_j)$$

$$P(x, z) \geqslant 0$$

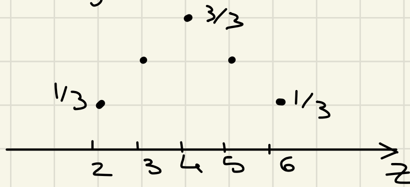
$$\sum_{i,j} P(x_i, z_j) = 1$$

**Densità Congiunta**

$$P(X = x_i) = \sum_{j=-\infty}^{+\infty} P(x_i, z_j)$$




$$P(z) =$$



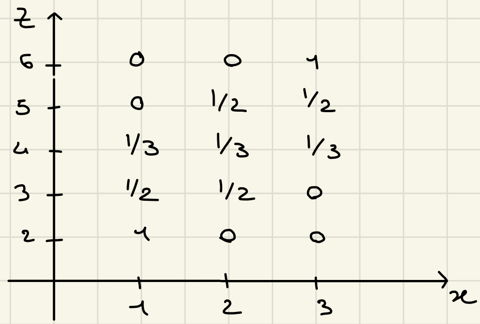

**Densità Marginale**

$$F_{xz} = P(X \leqslant x_i, Z \leqslant z_j) = 1/3$$

$x_i \to 2$, $z_j \to 3$

$$\boxed{P(x_i|_{z_j}) = \frac{P(x_i, z_j)}{P(z_j)}}$$

**Densità Condizionata**




Nel caso continuo:

$$f_{x,y}(x, y) = \frac{P(x \leqslant X \leqslant x + dx, y \leqslant Y \leqslant y + dy)}{dx\,dy}$$

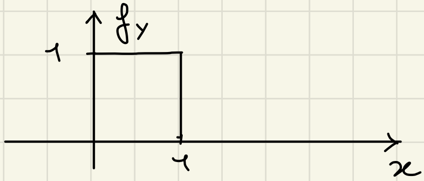
se indipend.

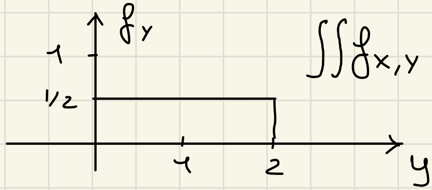
$$= \frac{P(x \leqslant X \leqslant x + dx)}{dx} \frac{P(y \leqslant Y \leqslant y + dy)}{dy} =$$

$$= f_X(x)\, f_Y(y)$$

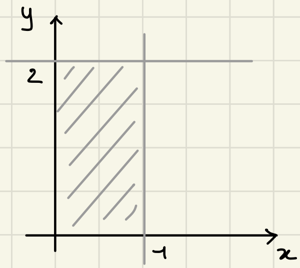
$x$ unif. $0 - 1$
$y$ unif. $0 - 2$

$$f_{x,y}(X, Y) = \underbrace{\mathrm{rect}\left(X - \tfrac{1}{2}\right)}_{f_x(X)} \underbrace{\tfrac{1}{2}\,\mathrm{rect}\left(\tfrac{Y-1}{2}\right)}_{f_y(Y)}$$




$$\iint f_{X,Y}(x,y)\,dx\,dy = 1$$




$$F_{X,Y}(x,y) = P(X \leq x,\ Y \leq y)$$

$$f_X(x) \underset{d^2/dxdy}{\overset{\iint}{\rightleftarrows}} F_X(x)$$

$$f_{X|Y}(x|_y) = \frac{P(x \leq X < x+dx|_{y \leq Y < y+dy})}{dx} =$$

$$= \frac{P(x \leq X < x+dx,\ y \leq Y < y+dy)}{P(y \leq Y < y+dy)\,dx} =$$

$$= \frac{f_{X,Y}(x,y)\,dx\,dy}{f_Y(y)\,dy\,dx} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_X(x) = \int f_{X|Y}(x|_y)\,f_Y(y)\,dy = \int f_{X,Y}(x,y)\,dy$$

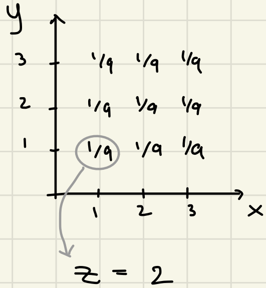
Ritornando al caso discreto (2 dadi a 3 facce):

$$P(\underbrace{D_1 + D_2}_{z})$$

sono indip.

$$P(\underset{D1}{x}, \underset{D2}{y}) = \underbrace{\underbrace{P(x)}_{1/3}\,\underbrace{P(y)}_{1/3}}_{1/9}$$

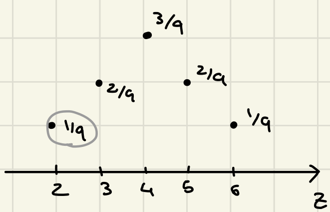
$$1 \leq z \leq 3$$
$$1 \leq y \leq 3$$




$P(z)$

$$z = D1 + D2$$
$$z = x + y$$
$$= -x + z$$

$$Z = \{2, 3, 4, 5, 6\}$$



$$P(z) = \sum_{i=-\infty}^{+\infty} P_{z|x_i}(z, x_i) = \sum_{i=-\infty}^{+\infty} P_{z|x_i}(z|x_i)\, P_x(x_i) =$$

$$= \sum_{i=-\infty}^{+\infty} P_{y|x_i}(\overbrace{z - x}^{y}|x_i)\, P_x(x_i) = \sum_{i=-\infty}^{+\infty} P_{x,y}(z-x_i, x_i)$$

$$= \sum_{i=-\infty}^{+\infty} P_x(x_i) \cdot P_x(-x_i+z)$$

$$\int f(x)\, g(z-x)\, dx = f(z) * g(z)$$

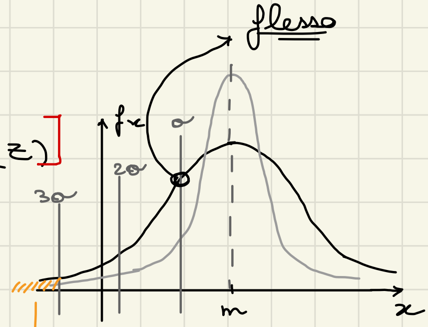
se indip.

convoluzione

$$\left[ P(x+y+z) = P(x) * P(y) * P(z) \right]$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-m)^2}{2\sigma^2}}$$

risultato generale della convoluzione



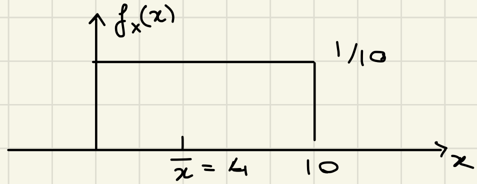

curva gaussiana

In Matlab: qfunc$(m-3\sigma)$

DENSITÀ e RIPARTIZIONI $\longrightarrow$ TRASFORMAZIONI

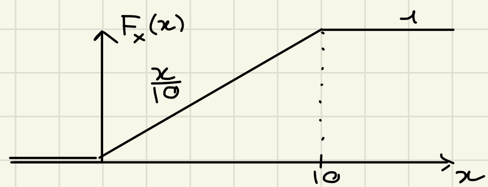
$$\boxed{y = x^2}$$

$x \in [0, 10]$
$y \in [0, 100]$




$$F_x(x) = P(x \leq x)$$

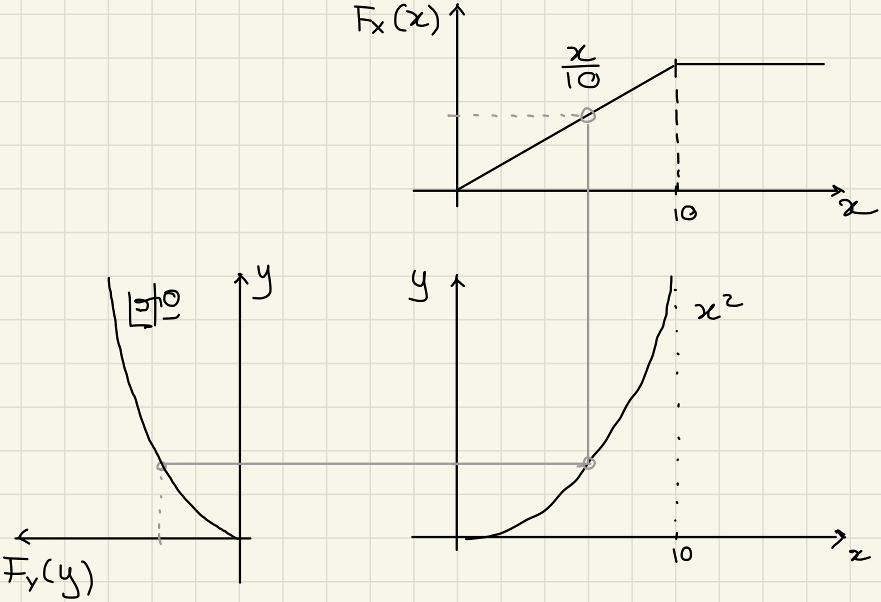
$$F_y(\bar{y} = 16) = F_y(\bar{x} = 4)$$




$y = g(x) \qquad y = x^2$
$x = g_I(y) \qquad x = \sqrt{y}$

$$F_x(x) = \int_0^x f_x(x)\, dx = \int_0^x \frac{1}{10}\, dx = \frac{x}{10}$$

$$F_x(x) = F_x(\sqrt{y}) = \frac{\sqrt{y}}{10}, \qquad 0 \leq y \leq 100$$

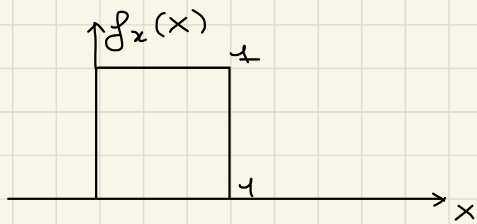


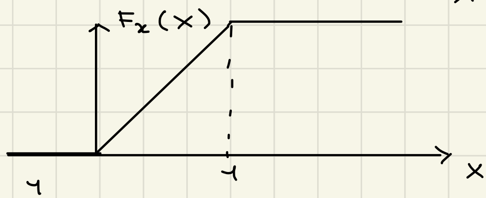

<u>es</u>: $y = 3x + 2 \quad x \in [0, 1]$

$y(x) \to 3x + 2$

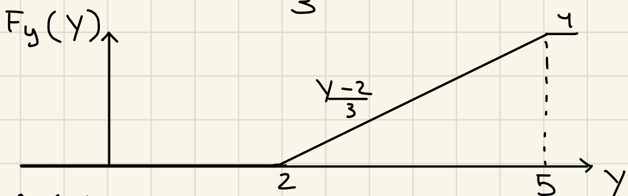
$x(y) \to \dfrac{y-2}{3}$

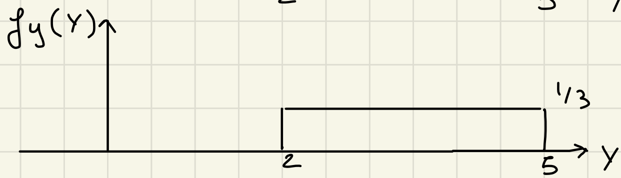
$F_y(Y) = F_x\left(\dfrac{Y-2}{3}\right)$
$= \dfrac{Y-2}{3}$



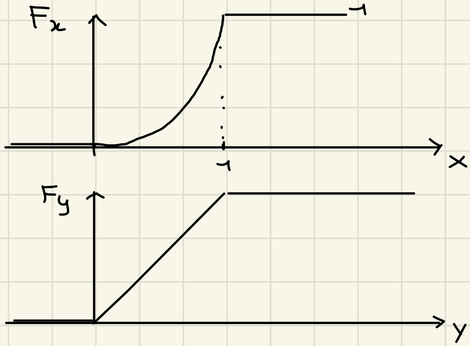

$F_x(X) = X$



es:

$F_x(X) = X^2$




$y = x^2 \quad x \in [0,1]$

$\hookrightarrow x = \sqrt{y} \quad y \in [0,1]$

$F_y(Y) = F_x(f^{-1}(y)) = (\sqrt{y})^2 = y$

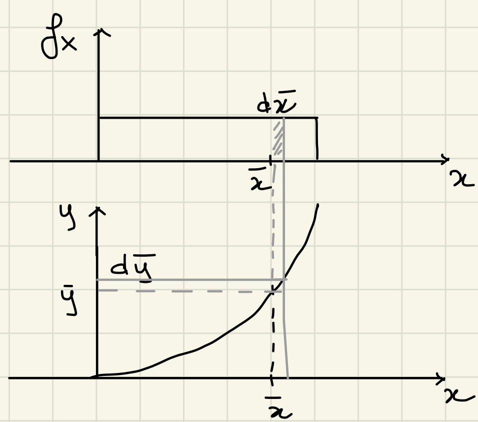
y var. cas. uniforme $0 \leq y \leq 1$

posso andare a ritroso per realizzare (su matlab) una variabile con densità di probabilità lineare a partire da una variabile a densità di probabilità uniforme.

$\boxed{y = x^2}$

$0 \leq x \leq 10$
$x$ v.c.u.

$\bar{x} \leq X < \bar{x} + d\bar{x}$
$\bar{y} \leq Y < \bar{y} + d\bar{y}$



$$x = g_I(y)$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

$$f_Y(\bar{y})\, d\bar{y} = f_X(\bar{x})\, d\bar{x}$$

$$\hookrightarrow f_Y(\bar{y}) = f_X(\bar{x}) \left| \frac{d\bar{x}}{d\bar{y}} \right| = \frac{f_X(g_I(\bar{y}))}{|g'(g_I(\bar{y}))|}$$

il modulo è dovuto al fatto che $\frac{dx}{dy}$ può essere positivo o negativo a seconda del l'andamento di $g$, ma $f$ deve essere sempre pos.

$$f_X(x) = \frac{1}{10} \operatorname{rect}\left(\frac{x-5}{10}\right)$$

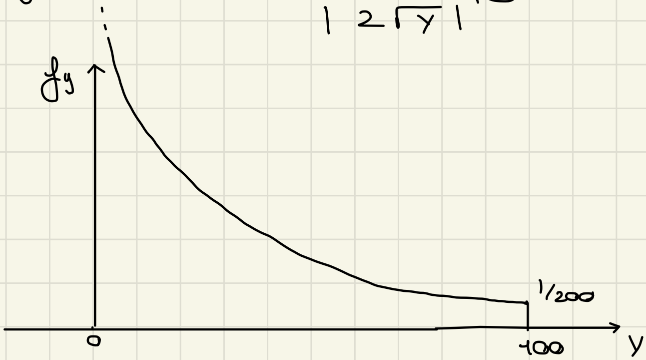
$$y = x^2 = g(x)$$

$$x = \sqrt{y} = g_I(y)$$

$$g'(x) = 2x$$

$$\Longrightarrow f_Y(y) = \frac{\frac{1}{10} \operatorname{rect}\left(\frac{\sqrt{y}-5}{10}\right)}{|2\sqrt{y}|} = \frac{1}{20\sqrt{y}} \operatorname{rect}\left(\frac{\sqrt{y}-5}{10}\right)$$

$$\operatorname{rect}\left(\frac{\sqrt{y}-5}{10}\right) = \operatorname{rect}\left(\frac{y-50}{10}\right)$$




$$\boxed{f_Y(y) = \frac{f_X(g_I(y))}{|g'(g_I(y))|}}$$

$$x_1 + x_2 + x_3 + \ldots = y \quad \text{v.c. Normale } N(\mu, \sigma)$$

$$y = \sum_k x_k$$

$\sigma$ deviazione quadratica standard

$\sigma^2$ varianza

$$x_1 \cdot x_2 \cdot x_3 \cdot \ldots = y = \prod_k x_k$$

Se $x_k \geq 0$

$$G(y) = \log(x_1 \cdot x_2 \cdot x_3 \dots) = \log(x_1) + \log(x_2) + \log(x_3) \dots$$

$z = \log(y)$ v. c. Gaussiana $\mathcal{N}(\mu_z, \sigma_z^2)$

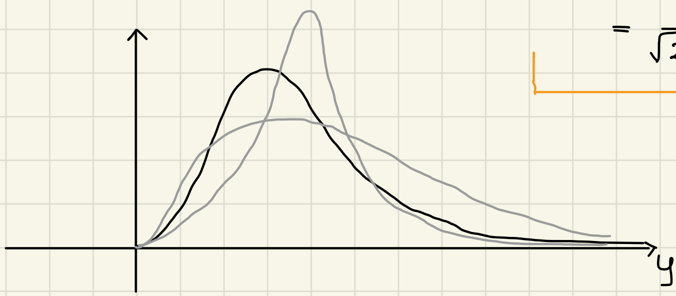
$$f_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-\frac{(z-\mu_z)^2}{2\sigma_z^2}} \qquad f_y(y) = ?$$

$$y = \exp(z) = g(z)$$
$$z = \log(z) = g_I(y)$$

$$\boxed{f_y(y) = \frac{f_z(g_I(y))}{|g'(g_I(y))|}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-\frac{(\log(y)-\mu_z)^2}{2\sigma_z^2}} \cdot \frac{1}{|y|}$$

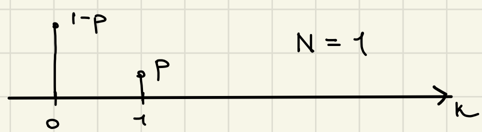
$y \geq 0$

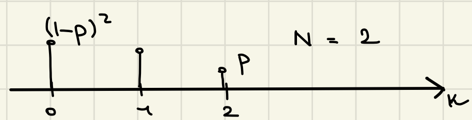



# FREQUENZA RELATIVA

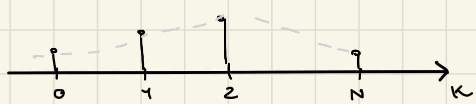
## Distribuzione Binomiale

$P_N(K)$ K successi
N prove

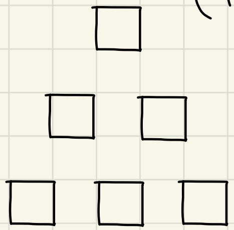



probabilità che esca "6" in N dadi



(triangolo di Tartaglia)



N = 1 $\qquad 1$

N = 2 $\qquad 1 \quad 1$

N = 3 $\qquad 1 \quad 2 \quad 1$

$1 \quad 3 \quad 3 \quad 1$

$1 \quad 4 \quad 6 \quad 4 \quad 1$

Coefficiente binomiale
$$\underbrace{\binom{N}{K} = \frac{N!}{K!\,(N-K)!}}_{\text{NCHOOSEK( )}}$$

$$\Longrightarrow \boxed{P_N(K) = \binom{N}{K} p^K (1-p)^{N-K}}$$

es: probab. di un "6" su 3 dadi

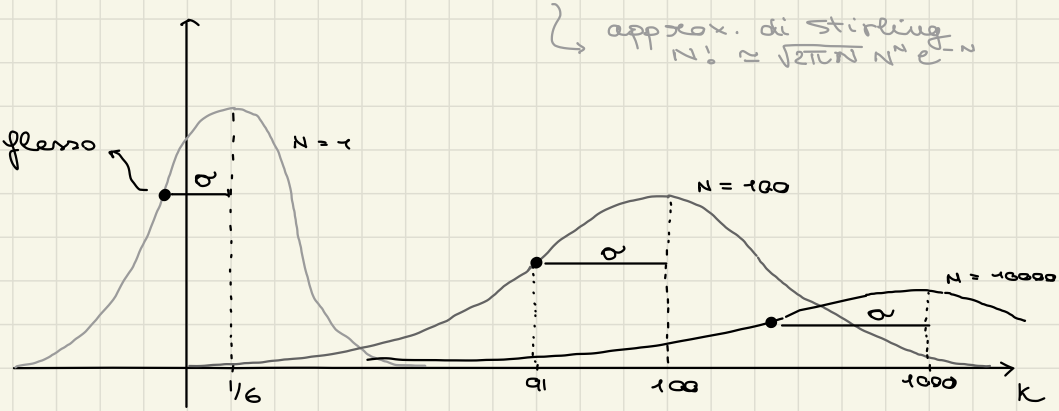
$$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = 3 \cdot \frac{25}{216}$$

$$P_N(K) = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{3!}{1!\,(2!)} \cdot \frac{25}{216} = 3 \cdot \frac{25}{216}$$

# Comportamento per N grande

$$\left[ P_N(k) \simeq \mathcal{N}(\mu = Np, \quad \sigma^2 = Np(1-p)) \right]$$

approx. di Stirling
$N! \simeq \sqrt{2\pi N}\, N^N e^{-N}$




$\boxed{N = 1}$

$$\mu = \frac{1}{6}$$

$$\sigma = \sqrt{1/6 \cdot 5/6} = \sqrt{\frac{5}{6}}$$

$\boxed{N = 600}$

$$\mu = 100$$

$$\sigma = \sqrt{N}\sqrt{\frac{5}{6}} \simeq 9{,}15$$

$91 \leq k \leq 109$
$78\,\%$

$\boxed{N = 60000}$

$$\mu = 10000$$

$$\sigma \simeq 91{,}5$$

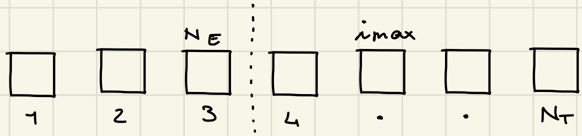
$N \to \infty$

Frequenza relativa $\nu = \dfrac{k}{N}$

$P_N(k) = \delta(\nu - p)$

Errore percentuale $\varepsilon_{\%} = \sigma/\mu = \sqrt{(1-p)/Np}$

# Esercizi:

- 


Dietro le carte ci sono dei numeri. Devo prendere la carta col numero più alto, fermandomi ad un certo punto. Quante carte devo vedere ($N_E$) per avere una buona stima dei valori per sapere a che punto fermarmi?

$$P_{N_E}(V) = \sum_{imax=1}^{N_T} P(V|imax)\, P(imax)$$

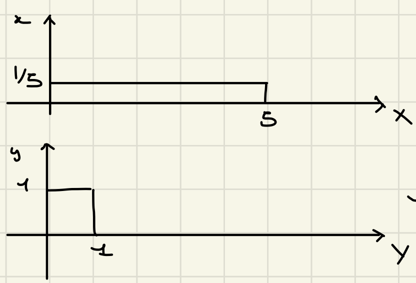
Vittoria

$$= \frac{1}{N_T} \sum_{imax=N_E+1}^{N_T} P(V|imax)$$

$$= \frac{1}{N_T} \sum_{imax=N_E+1}^{N_T} \frac{N_E}{(imax-1)}$$

$$= \frac{N_E}{N_T} \sum_{imax=N_E+1}^{N_T} \frac{1}{(imax-1)}$$

- $x$ u.c.u. $0 - 5$ $\qquad P(y > x) = ?$
  $y$ u.c.u. $0 - 1$




$$P(y>x) = \sum P(y>x|_{x=\bar{x}})\, P(x=\bar{x})$$

$$\int f(\bar{x})\, d\bar{x}$$

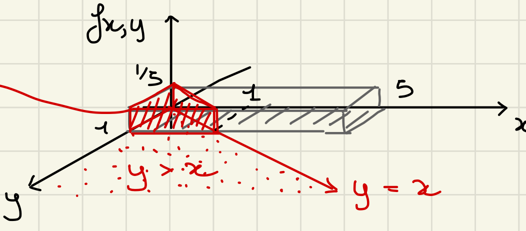
$$= \int_0^1 (1-\bar{x}) \cdot \frac{1}{5}\, d\bar{x}$$

$$= \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$$

Altro metodo:

se indip.

$$f_{xy}(x,y) = f_x(x) f_y(y) = \underbrace{\frac{1}{5}\,\mathrm{rect}\left(\frac{x-5/2}{5}\right)\cdot \mathrm{rect}\left(y-\frac{1}{2}\right)}$$




$$V = P(y > x) = B \cdot H = \frac{b \cdot h}{2} \cdot H = \frac{1 \cdot 1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

$$P(y>x) = \int_{y>x} f_{x,y}(x,y)\,dx\,dy$$

- $z = x + jy$    $x$ e $y$ v.c.i. (i.i.d) = variabili casuali indipendenti

$P(|z|) = ?$

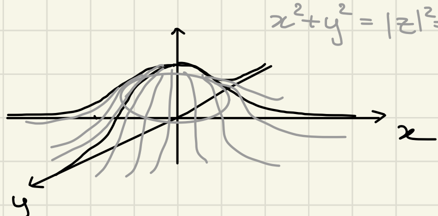
$$f_{x,y}(x,y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = N(0,\sigma^2)$$

$x^2+y^2 = |z|^2 = \rho^2$

$$F_\rho(\rho) = \int_0^\rho 2\pi\rho \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{2\pi\sigma^2} d\rho$$

$$= \int_0^\rho \frac{\rho}{\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} d\rho$$

$$= \left[-e^{-\frac{\rho^2}{2\sigma^2}}\right]_0^\rho = 1 - e^{-\frac{\rho^2}{2\sigma^2}}$$



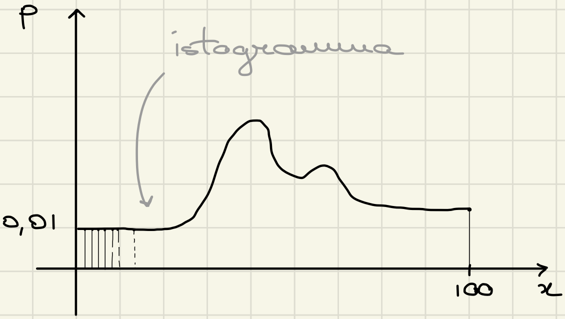

$x$ e $y$ Normali

$$\boxed{f_\rho(\rho) = \frac{\rho}{\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}} u(\rho)}$$ distribuzione di Rayleigh

$u(\rho)$ → funz. scalino

# MOMENTI

$P(x)$ $\quad x$ v.c.




$0 < x < 100$

$\hookrightarrow N_I = 100$

$f_x(x) \geq 0{,}01$

$acc_{\%} = \frac{\Delta p}{p} = 10\%$

accuratezza percentuale

$\Downarrow$

$N_{try} = ?$

$\sigma$ $\quad 1\cdot\sigma$

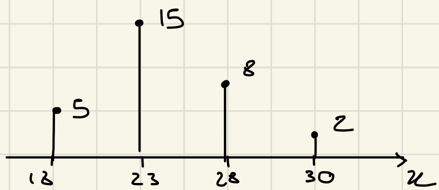
$$\frac{\Delta p}{p} = \frac{\sqrt{p(1-p)}}{\sqrt{N}\,p} \approx \frac{1}{\sqrt{N p}} = 0{,}1$$

$p << 1$

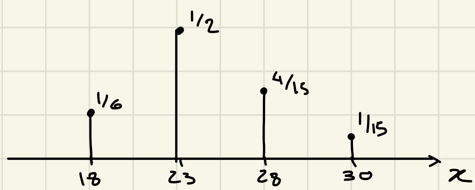
$$p = f_x(x)\,\Delta x \geq 0{,}01 \cdot 1$$

$$Np = 100 \rightarrow N = \frac{100}{p} = 10000$$

$x$ v.c $\Longrightarrow$ valore medio $\simeq$ media aritmetica




$\longrightarrow$




$$\boxed{m_N = \frac{1}{N}\sum_i x_i}$$ media aritmetica

$$= 18\cdot\frac{1}{6} + \frac{23}{2} + 28\cdot\frac{8}{30} + \frac{30}{15} = 23{,}9$$

$$\boxed{m_{1x} = \sum_{i=-\infty}^{+\infty} x_i \, p(x_i)} \quad \underline{\text{valor medio}} \overset{N \to +\infty}{=} m_N$$

momento non centrale di I° ordine $\Longrightarrow$ **BARICENTRO**

$$\boxed{m_{1x} = \int_{-\infty}^{+\infty} x \, f_x(x) \, dx} = E[x]$$

Expectation

operatore lineare

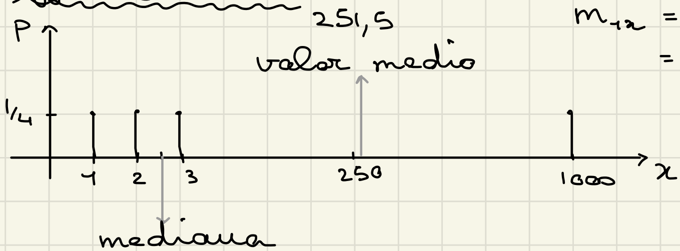
In una distribuzione simmetrica, $m_{1x}$ è anche il <u>centro di simmetria</u>

$$y = g(x)$$

$$m_{1y} = E[y] = \int y \, f_y(y) \, dy = \int g(x) \, f_x(x) \, dx$$

$$\boxed{E[g(x)] = \int g(x) \, f_x(x) \, dx}$$

## Effetto leva




$$m_{1x} = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 1000 \cdot \frac{1}{4} = 251{,}5$$

$\to$ poco rappresentativa

2,5 $\longrightarrow$ poco rappresentativa

V. M. $\longrightarrow$ DISTR. COMPATTA $\longleftarrow$ Mediana

$\downarrow$

DISPERSIONE

$$\boxed{\mu_{2x} = E\left[|m - x_i|^2\right]} \quad \underline{\text{varianza}} \quad (\sigma^2)$$

momento centrale di II° ordine

$$\mu_{2x} = E\left[x^2 + m_{1x}^2 - 2x\,m_{1x}\right]$$

linearità
$$= E\left[x^2\right] + m_{1x}^2 - 2\,E\left[x\,m_{1x}\right]$$

$$= E\left[x^2\right] + m_{1x}^2 - 2\,\overbrace{E\left[x\right]}^{m_{1x}}\,m_{1x} = E\left[x^2\right] - m_{1x}^2$$

# Momenti

| Centrali | Non Centrali |
|---|---|
| $\mu_{Nx} = E\left[(x - m_{1x})^N\right]$ | $m_{Nx} = E\left[x^N\right]$ |
| $\mu_{2x} = VAR$ | $m_{1x} = VM$ |

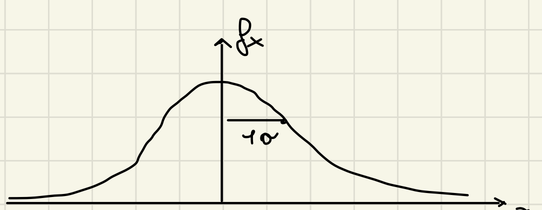
$$f(x) \longleftrightarrow F(f)$$
$$f'(x) \longleftrightarrow j2\pi f\,F(f)$$
$$j2\pi x\,f(x) \longleftrightarrow F'(f)$$

$$\boxed{\int x\,f_x(x)\,dx = \frac{1}{j2\pi}F'(f)\Big|_{f=0}}$$

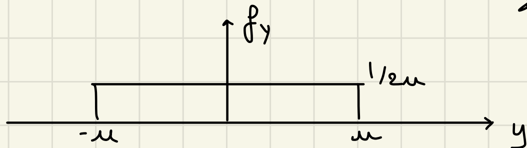
$$\int x^2 f_x(x)\,dx = k\,F''(f)\Big|_{f=0}$$

antitrasformando posso sempre calcolare l'integrale

Es: $x \sim N\left(0, \frac{1}{a^2}\right)$




$y$ v.c.u.
$m_y = 0$
$std(y) = 1$



$$\text{var}(y) = E[y^2] - m_{1y}^2 = \int_{-u}^{u} y^2 \frac{1}{2u}\, dy \quad f_Y(y)$$

$$= \frac{1}{2u}\left[\frac{y^3}{3}\right]_{-u}^{u} = \frac{1}{3}u^2$$

$$\text{std}(y) = \sqrt{\text{var}(y)} = 1$$

$$\frac{u}{\sqrt{3}} = 1 \implies u = \sqrt{3}$$

Due v.c. $x$ e $y$:

detto anche momento congiunto

- $E[xy] = r_{xy}$ correlazione non centrale
- $E[(x - m_{1x})(y - m_{1y})] = \sigma_{xy}$ covarianza centrale

$$= E[xy] - m_x m_y - \cancel{m_x m_y} + \cancel{m_x m_y} =$$

$$= E[xy] - m_x m_y$$

Se $x$ e $y$ sono indipendenti$^*$:

$$\sigma_{xy} = \iint xy\, f_{xy}(x,y)\, dx\, dy - m_x m_y$$

$$\overset{*}{=} \int x f_x(x)\, dx \int y f_y(y)\, dy - m_x m_y = 0$$

√ ×

INCORRELATE

$\sigma_{xy} = 0$ incorr.

$$\boxed{|\sigma_{xy}| \leq \sigma_x \sigma_y} \longrightarrow -1 \leq \frac{\sigma_{xy}}{\sigma_x \sigma_y} \leq 1$$

proprietà della covarianza

- $$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$ coeff. correlazione (lineare)

Se $x$ e $y$ sono linearmente dipendenti:

$$y = ax + b$$

$y = 2x \longrightarrow r_{xy} = 1$ correlazione totale

$y = -2x \longrightarrow r_{xy} < 0$ anticorrelazione

## Somma di v.c.

$z = x + y \qquad m_y = m_x = 0$

$$\begin{aligned} var(z) &= E[z^2] - \cancel{m_z}^{0} = E[(x+y)^2] = \\ &= E[x^2] + E[y^2] + 2E[\cancel{x \cdot y}] = \\ &= var(x) + var(y) \end{aligned}$$

$E[x]E[y] = 0$ poiché indipendenti

se $var(x) = var(y) = \sigma^2$

$$var(z) = \sigma^2 + \sigma^2 = 2\sigma^2$$

$\Longrightarrow \quad z = \sum_i^N x_i$ con $x_i$ v.c. indipend. (o incorrelate)

$$\left[ var(z) = \sum_i^N var(x_i) \right]$$

## Scalatura

$z = kx \qquad var(x) = \sigma_x^2$

$$\left[ \begin{aligned} var(z) &= E[k^2 x^2] - E[z]^2 = k^2 E[x^2] - k^2 E[x]^2 \\ &= k^2 \, var(x) \end{aligned} \right]$$

$$\Longrightarrow \quad z_N = \frac{1}{N}\sum_i^N x_i \quad \text{con } x_i \text{ i.i.d}$$

(indipendent identical distributed)

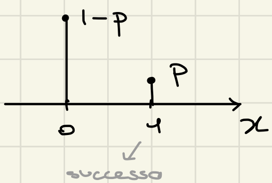
$$E[z_N] = \frac{1}{N}\sum_i^N \underbrace{E[x_i]}_{m_x} = m_x$$

$$var(z_N) = \frac{1}{N^2}\sum_i^N \sigma_{x_i}^2 = \frac{\sigma_x^2}{N}$$

$$\boxed{\lim_{N\to+\infty} P(|z_N - m_x| < \varepsilon) = 1 \quad \forall \varepsilon}$$

Probabilità $k$ successi $N$ prove

$x_1$ prob. $0, 1$ $\quad N = 1$ prova




$x_1$ v.c. binaria

$$m_x = \sum_i x_i \, p(x_i) = 0(1-p) + 1p = p$$

$$var(x) = \underbrace{\left[(1-p)\cdot 0^2 + p\cdot 1^2\right]}_{E[x^2]} - p^2 = p - p^2 = p(1-p)$$

$N > 1$ prove $\quad k = \sum_i^N x_i$

$$E[k] = \sum_i^N E[x_i] = Np$$

$$f_k(k) = ?$$

$$var(k) = Np(1-p)$$

# Teorema Limite Centrale

(classico) $$z = \frac{\left(\sum_{i=1}^{N} x_i\right) - N m_x}{\sqrt{N \sigma_x^2}}$$ con $x_i$ i.i.d. $\sigma_x^2 < \infty$

$N \to \infty \qquad z \sim \mathcal{N}(0, 1)$

(esteso) $$z = \frac{\sum_{i=1}^{N} (x_i - m_i)}{\sqrt{\sum_{i=1}^{N} \sigma_{x_i}^2}}$$

$N \to \infty \qquad f_z(z) \sim \mathcal{N}(0,1)$

Es. $x, y$ v.c. indip.

$$E[xy] = E[x]E[y] = m_x \cdot m_y$$

$\Downarrow$ incorrelate

$x = \cos t \qquad t$ v.c.u $0 - 2\pi$

$y = \sin t$

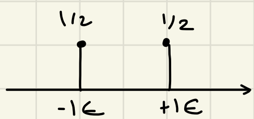
uniforme

$$r_{xy} = E[xy] = \frac{1}{2\pi} \int_0^{2\pi} \cos t \sin t \, f_t(t) \, dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(2t)}{2} \, dt = \varnothing$$

Es: testa o croce

$V_i$ : vincita




$m_x = 0$

$\mathrm{var}(x) = (-1)^2 \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$

$\sigma_x = 1$

$$z = \sum_{i=1}^{N} V_i \qquad \left.\begin{array}{l} m_z = 0 \\ var(z) = N \\ \sigma_z = \sqrt{N} \end{array}\right\} \mathcal{N}(0, \underset{var(z)}{N})$$

la varianza "esplode"

Es: $x$ v.c.u. $-\frac{1}{2} - \frac{1}{2}$ $\qquad z$ v.c.u. $-\frac{a}{2} - \frac{a}{2}$

$y = x + z \qquad x$ e $z$ indip.

1) "a" t.c. a) $r = 0,9$ e b) $r = 0,1$

$$var(z) = \int_{-a/2}^{a/2} \frac{1}{a} z^2 \, dz = \frac{1}{a} \left[ \frac{z^3}{3} \right]_{-a/2}^{+\frac{a}{2}} = \frac{a^2}{12}$$

$$r = \frac{E[xy] - \cancel{E[x]E[y]}^{\,0}}{\sqrt{\sigma_x^2 \sigma_y^2}} = \frac{E[x^2] - \cancel{E[xz]}^{\,0}}{\sqrt{\sigma_x^2(\sigma_x^2 + \sigma_z^2)}}$$

$$= \frac{1}{\sqrt{1 + \frac{\sigma_z^2}{\sigma_x^2}}} \Rightarrow \sqrt{1 + \frac{\sigma_z^2}{\sigma_x^2}} = \frac{1}{r}$$

$$\frac{\sigma_z^2}{\sigma_x^2} = \frac{1}{r^2} - 1$$

$$\left\{ \sigma_x^2 = \frac{1}{12} \right\} \qquad \sigma_z^2 = \frac{\sigma_x^2}{r^2}(1 - r^2)$$

$$\frac{a^2}{\cancel{12}} = \frac{1}{\cancel{12} r^2}(1 - r^2)$$

a) $a = 0,48$

b) $a = 9,95$

2) $f_{x,y}(x,y)$ ?

$$f_{x,y}(x,y) = \underbrace{f_{y|_x}(y|_x)}_{\frac{1}{a} rect\left(\frac{y-x}{a}\right)} \underbrace{f_y(x)}_{rect(x)}$$

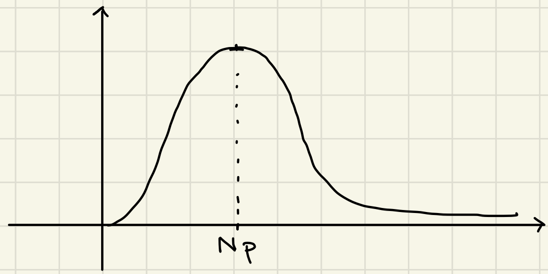
3) $f_y(y)$ ?

$$f_y(y) = f_x(x) * f_z(z)$$

# Distribuzione di Poisson

POISSON $\to$ CODE $\to$ n° di eventi in attesa
$\to$ tempo di attesa

Asintoticamente
$p \to 0$, $n \to \infty$




## Legge dei piccoli numeri.

frazione di tempo

1) probabilità per tempo $p = \nu \cdot dt$ (1 evento per volta)

2) eventi indipendenti $P_t(k)$

$\to$ $k = 0$ $\quad P_{t+dt}(0) = P_t(0)\,(1 - \nu \cdot dt)$

$$\cancel{P_t(0)} + dP_t(0) = \cancel{P_t(0)} - P_t(0)\,\nu\,dt$$

$$\boxed{\frac{dP_t(0)}{dt} = -\nu\,P_t(0)}$$

eq. diff. con sol.
$P_t(0) = e^{-\nu t}\,\underbrace{u(t)}_{\text{scalino}}$

$\to$ $k = 1$ $\quad P_{t+dt}(1) = P_t(0)\,\nu\,dt + P_t(1)\,(1 - \nu\,dt)$

$$\cancel{P_t(1)} + dP_t(1) = P_t(0)\,\nu\,dt + \cancel{P_t(1)} - P_t(1)\,\nu\,dt$$

$$\frac{dP_t(1)}{dt} = -\nu\,P_t(1) + \nu\,P_t(0)$$

$$P_t(1) = \nu t\,e^{-\nu t}\,u(t)$$

$\Delta t \cdot 0 \to t$

. . . . . $\quad P_{\Delta t}(k) = \dfrac{(\nu \Delta t)^k}{k!}\,e^{-\nu \Delta t}\,u(t)$

con $\nu \doteq$ eventi/secondo e $\lambda = \nu \Delta t \doteq$ eventi totali

$$\Longrightarrow \boxed{P_{\Delta t}(k) = \frac{\lambda^k}{k!} e^{-\lambda}}, \qquad \lambda = \nu \cdot \Delta t$$

La distribuzione di Poisson è una distribuzione binomiale portata al continuo.

Tempi di attesa – aspetto $t$ secondi per il 1° evento $T_1$

1) $F_{T_1}(t) = P(T_1 \leq t) = P(\text{almeno un evento}) =$
$= 1 - P_t(0) = 1 - e^{-\nu t}$

↳ è una probabilità cumulata

$\longrightarrow f_{T_1}(t) = \dfrac{dF_{T_1}(t)}{dt} = \nu e^{-\nu t}, \quad t \geq 0$

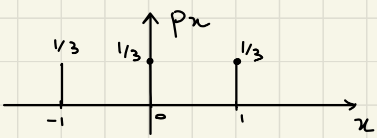
$t\nu = \lambda$
$d\lambda = \nu dt$

$$E[t] = \frac{1}{\nu} \qquad \sigma = \sqrt{\frac{1}{\nu}}$$

<u>Esercizi</u>:

- $x$, $e$ v.c.u. indip. $\{-1, 0, 1\}$

1) $p_x$, $m_x$, $\sigma_x^2$




$m_x = 0$

$$\sigma_x^2 = E[x^2] - \cancel{m_x^2} = \sum_i x_i^2 \, p_x(x_i) = \frac{2}{3}$$

2) $y = x + e \qquad r = ?$

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \qquad \sigma_x = \sqrt{\frac{2}{3}} \qquad \sigma_{xy} = E[xy] - \cancel{m_x m_y} = E[x^2 + xe]$$

$$E[x^2 + xe] = E[x^2] + E[xe] =$$
$$= \sigma_x^2 + E[x]E[e] = \sigma_x^2 = \frac{2}{3}$$

$$\sigma_y^2 = E[(x+e)^2] - \cancel{m_y^2} =$$
$$= E[x^2] + E[e^2] + 2\cancel{E[xe]} = \qquad \underbrace{E[x^2]} = \underbrace{E[e^2]}$$
$$= 2\sigma_x^2 = \frac{4}{3}$$

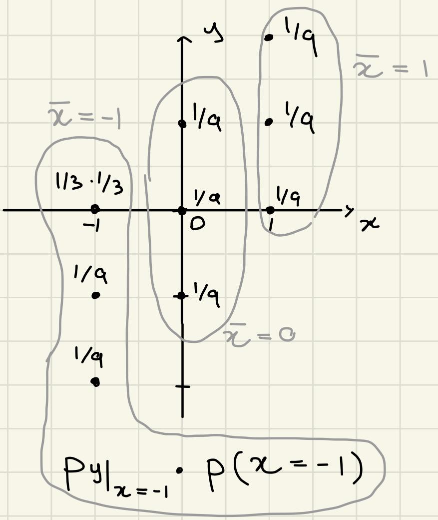
$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\cancel{2/3}}{\sqrt{\cancel{\tfrac{2}{3}}}\sqrt{\cancel{\tfrac{4}{3}}}} = \frac{1}{\sqrt{2}}$$

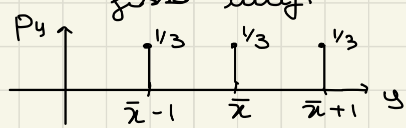
se $x$, $e$ indip
$var[y] =$
$= var[x] + var[e]$

3) $p_{xy}(x,y) = ?$

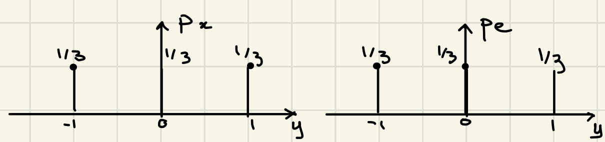
$$p_{xy}(x,y) = p_{x|y}(x|y)\, p_y(y) = p_{y|x}(y|x)\, p_x(x)$$

Poiché $y = x + e$ blocco $x$ così che
$$y = \bar{x} + e$$
$\bar{x}$ ↓ fisso, $e$ ↓ unif.

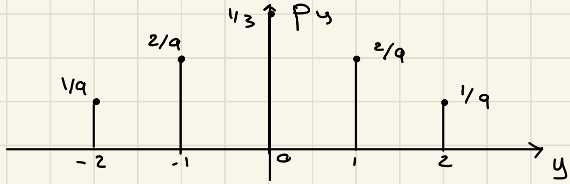



4) $p_y(y) = ?$ $\qquad y = x + e \longrightarrow p_y(y) = p_x(y) * p_e(y)$



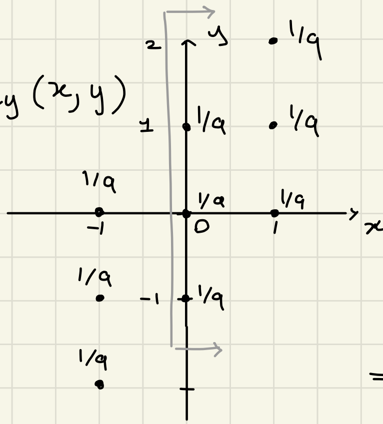
$$p_x(y) * p_e(y) = \sum_{n=-\infty}^{+\infty} p_x(n)\, p_e(y-n)$$

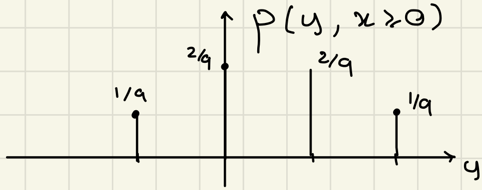



5) $p(y|_{x\geq 0}) = ?$  $p(y|_{x\geq 0}) = \dfrac{p(y, x\geq 0)}{p_x(x\geq 0)}$

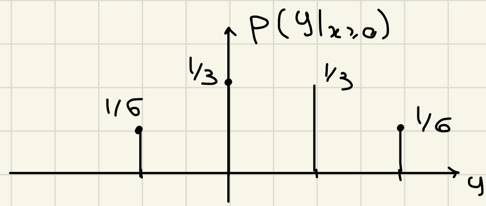
$\dfrac{2}{3}$

$p_{xy}(x,y)$

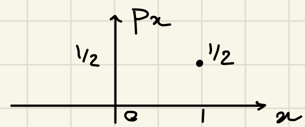


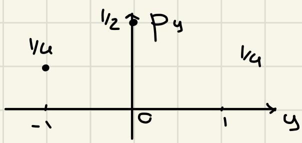

$\Longrightarrow$




- x v.c.u. binaria {0, 1}

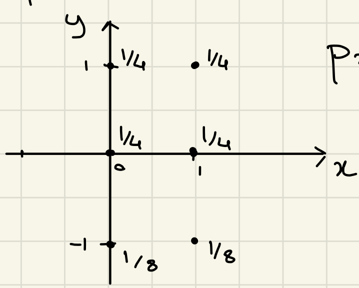



y v.c.




x e y indip.

1) $p_{xy}(x,y) = ?$



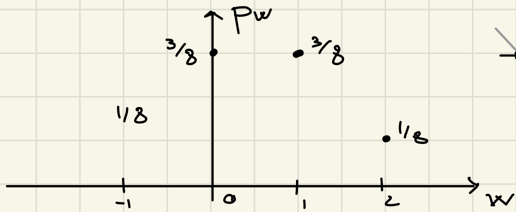

$p_{xy}(x,y) = p_x(x)\, p_y(y)$

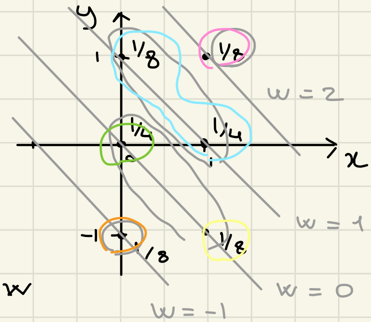
xx indip.

2) $w = x + y$ $\qquad P_{wq}(w, q) = ?$

$q = x \cdot y$

(w) $\quad y = -x + w$

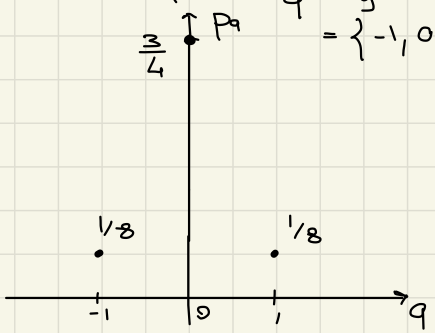


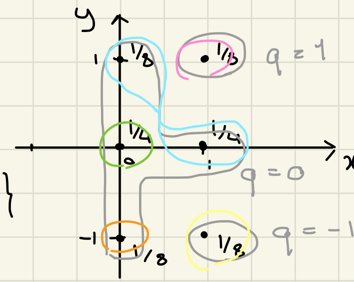




(q) $\quad q = x \cdot y$

$x = 0 \rightarrow q = 0$

$x = 1 \rightarrow q = y = \{-1, 0, 1\}$



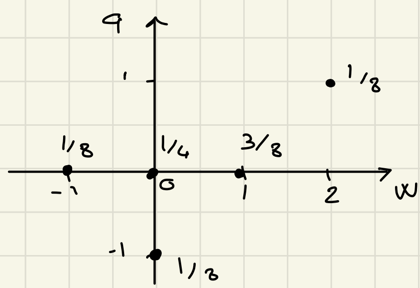




$q = 0 \begin{cases} w = -1 \longrightarrow 1/8 \\ w = 0 \longrightarrow 1/4 \\ w = 1 \quad \cdot\cdot \quad 3/8 \\ w = 2 \qquad 0 \end{cases}$

$q = -1 \begin{cases} w = -1 \qquad 0 \\ w = 0 \qquad 1/8 \\ w = 1 \qquad 0 \\ w = 2 \qquad 0 \end{cases}$

$q = 1 \begin{cases} w = -1 \qquad 0 \\ w = 0 \qquad 0 \\ w = 1 \qquad 0 \\ w = 2 \longrightarrow 1/8 \end{cases}$

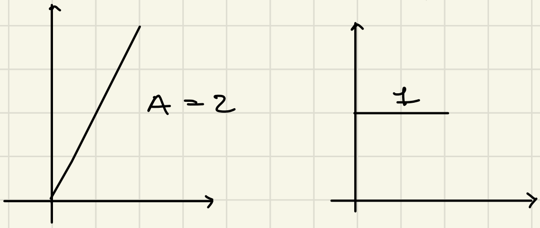
$P_{wq}(w, q)$



• $x$ v.c. $f_x(x) = A x \, \mathrm{rect}(x - \frac{1}{2})$

$y$ v.c. $f_y(y) = \mathrm{rect}(y - \frac{1}{2})$ indip.

1) $m_x$, $m_y$, $\sigma_x^2$, $\sigma_y^2$, $A$




$$m_x = E[x] = \int_0^1 x \, 2x \, dx = \left[\frac{2}{3} x^3\right]_0^1 = \frac{2}{3}$$

$$m_y = E[y] = \frac{1}{2}$$

$$\sigma_x^2 = E[x^2] - m_x^2 = \int_0^1 x^2 \cdot 2x \, dx - \frac{4}{9} =$$
$$= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\sigma_y^2 = \frac{1}{12}$$

quando la distribuzione è uniforme, la varianza è sempre $\frac{\text{intervallo}^2}{12}$

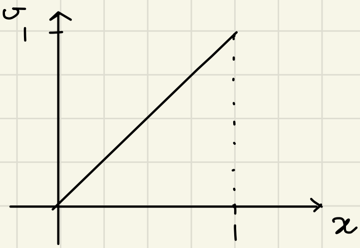
2) $v = x \cdot y$, $m_v$ e $\sigma_v^2$

xx sono indip

$$m_v = E[v] = E[xy] = E[x]E[y] = \frac{1}{3}$$

$$\sigma_v^2 = E[v^2] - m_v^2 = E[x^2 y^2] - \frac{1}{9}$$
$$= E[x^2]E[y^2] - \frac{1}{9}$$
$$= \frac{1}{2} \cdot \frac{1}{3} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18} = \frac{1}{18}$$

3) $f_{v|x}(v|_x)$ $v = Xy$ cioè $v$ è un rettangolo alto $X$ che si estende da $0 - 1/X$




$$f_{V|X}(v|x) = \frac{f_Y(y)}{\left|\frac{dv}{dy}\right|} \qquad y = \frac{v}{X}$$

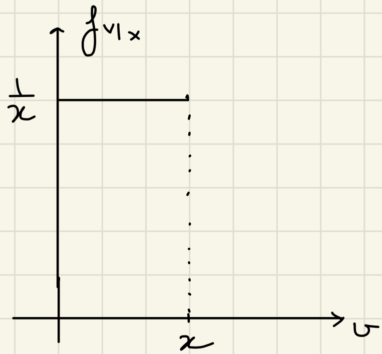
$$= \frac{\text{rect}\,(y - 1/2)}{X}$$

$$= \frac{1}{X}\,\text{rect}\left(\frac{v}{X} - \frac{1}{2}\right)$$

$$\Longrightarrow \quad f_{V|X}(v|x) = \frac{1}{x}\,\text{rect}\left(\frac{v - x/2}{x}\right) \qquad x \in (0,1)$$

$$\downarrow$$

$$v \in (0, x)$$

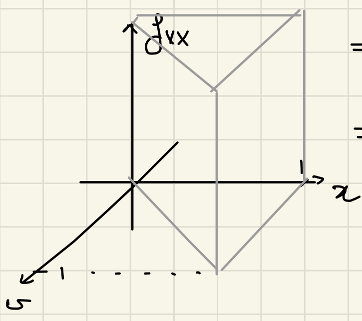



3) $f_{VX}(v, x)$

$$f_{VX}(v,x) = f_{V|X}(v|x)\, f_X(x)$$

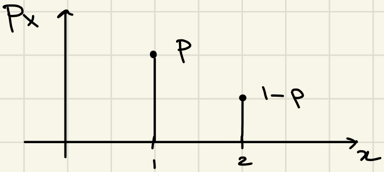
$$= \frac{1}{x}\,\text{rect}\left(\frac{v}{x} - \frac{1}{2}\right) 2x\,\text{rect}\left(x - \frac{1}{2}\right)$$

$$= 2\,\text{rect}\left(\frac{v}{x} - \frac{1}{2}\right) \text{rect}\left(x - \frac{1}{2}\right)$$




4) $f_V(v) = \int f_{VX}(v, x)\, dx = \int$

- $x$ v.c. binaria $\quad p_x(x) = p\,\delta(x-1) + (1-p)\,\delta(x-2)$

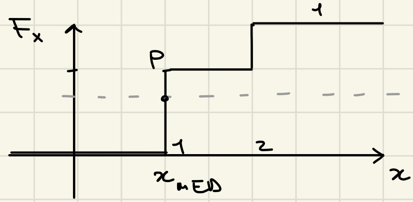



1) $m_x$, $\sigma_x^2$, $F_x(x)$, mediana

$$m_x = \sum_i x_i\, p_x(x_i) =$$
$$= 1 \cdot p + (1-p) \cdot 2$$
$$= p + 2 - 2p = 2 - p$$

$$\sigma_x^2 = \sum_i x_i^2\, p_x(x_i) - m_x^2$$
$$= p + 4(1-p) - (2-p)^2$$
$$= p + \cancel{4} - \cancel{4p} - (\cancel{4} + p^2 - \cancel{4p})$$
$$= p(1-p)$$

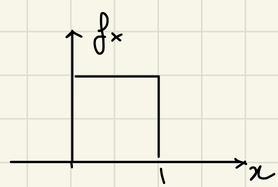
$$F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx =$$



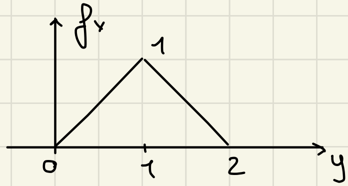

mediana $= x_{MED}$ $\quad F_x(x_{MED}) = 0{,}5$

$\hookrightarrow$ se $p < 0{,}5 \longrightarrow x_{MED} = 2$
se $p > 0{,}5 \longrightarrow x_{MED} = 1$

- $x$ v.c. uniforme 0 − 1



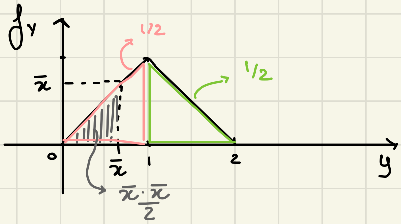

$y$ v.c.




$P(y > x)$ ?

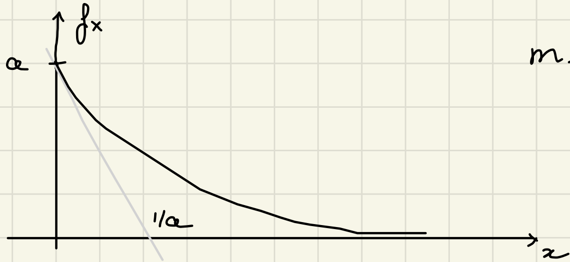
$$P(y > x) = \int P(y > x|_{x=\bar{x}})\, f_x(\bar{x})\, d\bar{x} =$$

$$= \int_0^1 P(y > \bar{x})\, d\bar{x} = \int_0^1 \left( \frac{1}{2} + \frac{1}{2} - \frac{\bar{x}^2}{2} \right) d\bar{x}$$




$$= \left[ \bar{x} - \frac{\bar{x}^3}{6} \right]_0^1 = \frac{5}{6}$$

- $x$ v.c. $f_x(x) = a\, e^{-ax} u(x)$




$m_x, \sigma_x^2$ ?

$$m_x = E[x] = \int_{-\infty}^{+\infty} x\, f_x(x)\, dx = \int_{-\infty}^{+\infty} x\, a e^{-ax} u(x) dx$$

$$\downarrow$$

$\int f(t)dt = F(f)|_{f=0}$

$F'(f) = j2\pi x f(x)$

$\frac{1}{j2\pi} F'(f) \Big|_{f=0}$ dove $F(f)$ è la trasformata di Fourier di $f(x)$

$$F(f) = \frac{1}{1 + j2\pi f/a}$$

$$E[x] = \frac{1}{j2\pi} \frac{j2\pi/a}{(1 + j2\pi f/a)^2} \Bigg|_{f=0} = \frac{1}{a}$$

$$\sigma_x^2 = E[x^2] - m_x^2 = \int_0^\infty x^2 a e^{-ax} dx - \frac{1}{a^2}$$

$$\downarrow$$

$$\left(\frac{1}{j2\pi}\right)^2 F''(f) \Big|_{f=0} = \frac{2}{a^2}$$

$$\text{var}(x) = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$